Journal of Pure and Applied Algebra 88 (1993) 143–154
North-Holland

Torsion in profinite completions

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Communicated by A. Camina
Received 15 December 1992

For Karl Gruenberg on his 65th birthday

1. Introduction

In [1] the question was raised whether the profinite completion of a torsion-free residually finite group is necessarily torsion-free. Subsequently, counter-examples were discovered by Evans [2], and more recently, Lubotsky [4] has shown that there is a finitely generated torsion-free residually finite group whose profinite completion contains a copy of every finite group.

In this paper we study the situation more closely for soluble groups. The major result of the paper is the construction of finitely generated centre by metabelian counter-examples in Section 3.

We begin in Section 2 by establishing some positive results showing that there are no counter-examples amongst abelian groups, finitely generated abelian by nilpotent groups or soluble minimax groups. In Section 3 we first establish counter-examples which are nilpotent of class 2: these are necessarily infinitely generated. Then we give the promised centre by metabelian examples.

It is interesting to note that the dichotomy arising here between abelian by nilpotent groups and centre by metabelian groups is the one which arises in the work of Hall [3].

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2. The set of torsion elements in profinite completions

In this section we explore circumstances in which there is a close relationship between torsion in the profinite completion $\widehat{G}$ of a soluble group G and torsion in G. It is reasonable to restrict attention to the residually finite case, so that G is naturally embedded in $\widehat{G}$. For any group G we write $t(G)$ for the set of torsion elements of G. If G is residually finite we have $t(G) \subseteq t(\widehat{G})$, and if there is a bound on the orders of the torsion elements of G then $t(G) = \{g \in G \mid g^e = 1\}$ for some integer $e > 0$, so that the closure $\overline{t(G)} \subseteq t(\widehat{G})$. Our first result, a slight generalization of a result shown to us by Zoé Chatzidakis, establishes the reverse inequality when G is abelian.

Proposition 2.1. *Let G be a residually finite abelian group. Then the torsion subgroup of $\widehat{G}$ lies in the closure of the torsion subgroup of G.*

Proof. Let h be a torsion element of $\widehat{G}$, of order n, say. We write G additively. We must show that for every subgroup U of finite index in G there is a torsion element g_U of G such that $U + g_U$ is the image of h in G/U. Choose $g \in G$ such that h maps to $U + g$ in G/U. Suppose $ng \not\in nU$. Since G/nU has finite exponent it is a direct sum of finite cyclic groups and so is residually finite. Therefore we can find a subgroup V of finite index in G such that $nU \leq V$ and $ng \not\in V$. Let $V + g_1$ be the image of h in G/V. Then we have $ng_1 \in V$ and $g_1 - g \in U$. It follows that

$$n(g_1 - g) \in nU \leq V$$

so that $ng \in V$, a contradiction. We conclude that there is an element $u \in U$ such that $ng = nu$. We therefore have $U + g_U = U + g$ and $ng_U = 0$, where $g_U = g - u$. $\square$

Our next objective is to prove a result similar to Proposition 2.1 for finitely generated abelian by nilpotent groups. Let G be finitely generated group with an abelian normal subgroup M such that $Q = G/M$ is nilpotent. The action of G on M by conjugation endows M with a natural $\mathbb{Z}Q$-module structure, and the profinite topology on G induces the $\mathbb{Z}Q$-module profinite topology on M.

Lemma 2.2. *Let $n \geq 1$ be a natural number and $g \in G$ an element such that $g^n \in M$. Then $M^{q^{n-1} + \cdots + q + 1}$ is closed in the profinite topology on M, where q denotes the image of g in Q.*

Proof. Since q is an element of finite order and Q is finitely generated nilpotent, it follows that $Q_1 = C_Q(q)$ has finite index in Q. The profinite topology on M as $\mathbb{Z}Q$-module coincides with the profinite topology as $\mathbb{Z}Q_1$-module. Therefore it suffices to show that $M^{q^{n-1}+\cdots+q+1}$ is closed in the profinite $\mathbb{Z}Q_1$-module topology. Note that $M^{q^{n-1}+\cdots+q+1}$ is a $\mathbb{Z}Q_1$-submodule of M because Q_1 centralizes q. Therefore the result follows since $M/M^{q^{n-1}+\cdots+q+1}$, being a finitely generated module for a finitely generated nilpotent group, is residually finite from the classical result of Hall [3]. $\square$

Lemma 2.3. *If Q is a finitely generated nilpotent group then $t(Q) = t(\widehat{Q})$.*

Proof. Note that $t(Q)$ and $t(\widehat{Q})$ are normal subgroups of Q and $\widehat{Q}$ both yielding torsion-free quotients. The result now follows from the fact that the profinite completion of a finitely generated torsion-free residually finite nilpotent group is again torsion-free. $\square$

Theorem 2.4. *If G is a finitely generated abelian by nilpotent group then $t(\widehat{G}) = \overline{t(G)}$.*

Proof. Let M be an abelian normal subgroup such that $Q = G/M$ is nilpotent. First note that, because G is finitely generated and Q is nilpotent, the torsion elements of M have bounded order, and hence

$$\overline{t(G)} \subseteq t(\widehat{G}) .$$

Now suppose that h is an element of $\widehat{G}$ such that $h^n = 1$ for some integer $n \geq 1$. To show that $h \in \overline{t(G)}$ it will suffice to show that for every normal subgroup K of finite index in G there is a torsion element g_K of G such that Kg_K is the image of h in G/K. Let L_K be the image of h in G/K and let $g_1, g_2, \ldots$ be a sequence of elements of L_K which converges to h in the profinite topology. By Lemma 2.2, $t(Q) = t(\widehat{Q})$, and so we may assume that every g_i has the same image q in Q. We may now write $h = g_1 m$ and $g_i = g_1 m_i$ for some $m \in \widehat{M}$ and some sequence (m_i) of elements of M. Let $M = U_0 \geq U_1 \geq U_2 \cdots$ be a base of neighbourhoods of 1 in the profinite $\mathbb{Z}Q$-module topology on M. Since the sequence (m_i) is convergent in $\widehat{M}$ to $m = g_1^{-1}h$, for any i there is a j such that for all $k,l \geq j$ we have $m_k \equiv m_l$ modulo U_i. Since also $(g_1 m_i)^n$ converges to 1 as $i \to \infty$, by passing to a subsequence we may assume that for all $i \geq 0$ and all $l \geq i$ we have

$$m_i \equiv m_l \quad \text{modulo } U_i ,$$

and

$$(g_1 m_i)^n = g_1^n m_i^{q^{n-1}+\cdots+1} \in U_i .$$

In particular it follows that $g_1^n \in U_i M^{q^{n-1}+\cdots+1}$ for all i, and since $M^{q^{n-1}+\cdots+1}$ is closed by Lemma 2.2, it follows that $g_1^n \in M^{q^{n-1}+\cdots+1}$. Choose m_0 so that

$g_1^n = m_0^{q^{n-1}+\cdots+1}$. It is now clear that $Kg_K = Kg_1$ and $g_K^n = 1$, where $g_K = g_1 m_0^{-1}$. This proves the theorem. □

Our final result in this section concerns torsion in profinite completions of minimax groups. We begin with two elementary lemmas which are no doubt well known.

Lemma 2.5. *Let G be a minimax group and let A be a normal subgroup of G which is closed in the profinite topology on G. Then the subspace topology on A induced by the profinite topology on G coincides with the profinite topology on A.*

Proof. It will suffice to show that every normal subgroup B of finite index in A is closed in the profinite topology on G. Now if B has index n in A then B contains the subgroup A_n generated by all nth powers in A, and A_n is normal in G. Since A is closed the group G/A is residually finite, and so has no non-trivial quasicyclic subgroups. However, A/A_n is finite, and thus G/A_n can have no non-trivial quasicyclic subgroups. Therefore G/A_n is residually finite, and we conclude that both A_n and B are closed in the profinite topology on G. □

Lemma 2.6. *If A is a torsion-free abelian minimax group then the group $\widehat{A}/A$ is torsion-free and divisible.*

Proof. The result follows since it is easy to check that

$$n\widehat{A} + A = \widehat{A} \quad \text{and} \quad n\widehat{A} \cap A = nA$$

for all positive integers n. □

Theorem 2.7. *Let G be a residually finite soluble minimax group. Then every finite subgroup of $\widehat{G}$ is conjugate to a subgroup of $t(G)$. Consequently, $t(\widehat{G}) = \overline{t(G)}$.*

Proof. The second statement follows from the first since $\overline{t(G)}$ is clearly closed with respect to conjugation in $\widehat{G}$ and since the elements of $t(G)$ have bounded order.

We shall prove the first statement by induction on the length of a series of normal subgroups of G each of whose factors is either an abelian torsion group or an abelian torsion-free group. The result clearly holds if there is such a series of length 1. Assume that there is a series of length greater than 1, and let A be the first non-trivial subgroup in this series. Replacing A if necessary by its closure in G in the profinite topology we may assume that $Q = G/A$ is residually finite. If A is a torsion group then it is finite, and the result follows immediately by induction. We may therefore assume that A is torsion-free.

Let H be a finite subgroup of $\widehat{G}$: we must show that H is conjugate to a subgroup of G. By induction, the image of H in $\widehat{Q}$ is conjugate to a subgroup of Q. Now Lemma 2.5 shows that the closure of A in $\widehat{G}$ is naturally isomorphic to $\widehat{A}$. Moreover, the groups $\widehat{Q}$ and Q can be identified with $\widehat{G}/\widehat{A}$ and $G\widehat{A}/\widehat{A}$, respectively. Thus the inductive hypothesis shows that $H^g\widehat{A}$ is contained in $G\widehat{A}$ for some $g \in \widehat{G}$. Replacing H by H^g we may assume that $H \leq G\widehat{A}$. Since both G and $\widehat{A}$ normalize A it follows that H normalizes A. We can view $\widehat{A}$ as an H-module and A as an H-submodule via the action of H by conjugation. For each $h \in H$ let $g_h \in G$ and $x_h \in \widehat{A}$ be elements such that $h = g_h x_h$. Although x_h is not uniquely determined by h, its image in $\widehat{A}/A$ is uniquely determined because $G \cap \widehat{A} = A$. In fact, the function $\delta : H \to \widehat{A}/A$ defined by $h \mapsto Ax_h$ is a derivation. Now according to Lemma 2.6 $\widehat{A}/A$ is torsion-free and divisible, and since H is finite it follows that the first cohomology group $H^1(H, \widehat{A}/A)$ is zero. Thus δ is an inner derivation and so there is an element $b \in \widehat{A}$ such that $\delta(h) = Ab^{-1}h^{-1}bh$ for each $h \in H$. This implies that h^b belongs to G for each $h \in H$. Thus H^b is the required conjugate of H. □

3. The counter-examples

Throughout this section, let p and q be prime numbers. We construct groups G which satisfy the following:

(i) G is torsion-free and residually finite, and in the case $p = q$, G is residually a finite p-group.

(ii) the profinite completion of G contains an element of order q;

(iii) if $p = q$ then the pro-p completion of G contains an element of order p.

We construct two kinds of counter-example. In the first, described in Examples 3.2 and 3.3, G is nilpotent of class 2, with $\zeta(G)$ free abelian of finite rank and $G/\zeta(G)$ free abelian of infinite rank. The second kind, described in Theorem 3.9, is finitely generated and centre by metabelian, again with $\zeta(G)$ being free abelian of finite rank and now with $G/\zeta(G)$ isomorphic to a restricted wreath product of a free abelian group of finite rank by an infinite cyclic group.

For the nilpotent example, we fix the following notation. Let n be a natural number and let Z be a free abelian group of rank n. Let

$$Z = Z_0 \geq Z_1 \geq Z_2 \geq \cdots$$

be a descending chain of subgroups of finite index in Z.

Proposition 3.1. *There exists a class-2 nilpotent group G with centre Z such that*

(i) *for each k there is a normal subgroup N of finite index in G with $N \cap Z = Z_k$; and*

(ii) *every normal subgroup of finite index in G contains Z_k for some k.*

Proof. For each i let $\{z_{i,1}, \ldots, z_{i,n}\}$ be a set of generators of Z_i. Let A, B be free abelian groups on the sets of generators $\{a_{i,j} \mid i \geq 0,\ 1 \leq j \leq n\}$ and $\{b_{i,j} \mid i \geq 0,\ 1 \leq j \leq n\}$, respectively. Let G be the group generated by A and B subject to the relations

$[a_{i,j}, b_{k,l}] = 1$ if $(i, j) \neq (k, l)$,
$[a_{i,j}, b_{i,j}] = z_{i,j}$, and
Z is central.

Clearly these relations define a class-2 nilpotent group with centre Z. To check (i), choose $k \geq 0$ and let N_1 be the normal closure of the $a_{i,j}$ and $b_{i,j}$ with $i \geq k$. Then N_1 meets Z in Z_k. Moreover, N_1Z/N_1 is finite and G/N_1 is finitely generated and nilpotent and so residually finite. It follows that G has a normal subgroup N of finite index which meets N_1Z in N, and clearly N meets Z in Z_k. Now we check (ii). Suppose that N is a normal subgroup of finite index in G which does not contain any Z_i. Since G/N is finite we can find an index j and a subsequence $z_{n_0,j}$, $z_{n_1,j}$, $z_{n_2,j}, \ldots$ of $(z_{i,j}\colon i \geq 0)$ such that

$$N \neq Nz_{n_0,j} = Nz_{n_1,j} = Nz_{n_2,j} = Nz_{n_3,j} = \cdots .$$

Again, the finiteness of G/N ensures that there exist distinct k, l such that $Na_{n_k,j} = Na_{n_l,j}$. But then

$$N \neq Nz_{n_k,j} = N[a_{n_k,j}, b_{n_k,j}] = N[a_{n_l,j}, b_{n_k,j}] = N ,$$

and this is a contradiction. This completes the proof of (ii). $\square$

The example constructed in Proposition 3.1 is necessarily torsion-free because it is nilpotent and has torsion-free centre. If in addition the following condition is satisfied,

$$\bigcap_{i \geq 0} Z_i = 1 ,$$

then it is residually finite. What the proposition ensures is that the closure of Z in $\widehat{G}$ is the completion

$$\varprojlim Z/Z_i$$

with respect to the filtration by the Z_i. As a very simple application we note the following.

Example 3.2. *Let p and q be distinct primes. There exists a class-2 nilpotent group G with infinite cyclic centre which is residually a finite p-group but such that $\widehat{G}$ contains an element of order q.*

Proof. Take Z to be an infinite cyclic group, set $Z_0 = Z$ and $Z_i = Z^{p^i q}$ for each $i \geq 1$ and construct G as in Proposition 3.1. Note that the completion $\varprojlim Z/Z_i$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}/q\mathbb{Z}$, where $\mathbb{Z}_p$ denotes the p-adic integers.

Example 3.3. *Let p be a prime. There exists a class-2 nilpotent group G, with centre a free abelian group of rank 2, which is residually a finite p-group but such that $\widehat{G}_p$ contains an element of order p.*

Proof. Here, it is convenient to describe Z as a subgroup of $\mathbb{Z}_p$, and so we use additive notation for Z. As an abstract additive group, $\mathbb{Z}_p$ is torsion-free of infinite rank. Let Z be any free abelian subgroup of rank 2. Now set $Z_0 = Z$ and for $i \geq 1$, Z_i to be $p^i\mathbb{Z}_p \cap pZ$. Since Z/Z_i is a p-group for all i, it follows that the group G of Proposition 3.1 is actually residually a finite p-group. Finally, note that the completion $\varprojlim Z/Z_i$ is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}/p\mathbb{Z}$ and so $\widehat{G}_p$ contains an element of order p. $\square$

We now fix the notation for the centre by metabelian examples. Let n be a positive integer, to be specified more precisely later. Let A denote the ring $\mathbb{Z}^n = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, let R denote the Laurent polynomial ring $A[x, x^{-1}]$ over A, and let U be the subgroup $\langle x \rangle$ of units of R. Define H to be the matrix group

$$H = \begin{pmatrix} 1 & R & R \\ 0 & U & R \\ 0 & 0 & 1 \end{pmatrix}.$$

This is a finitely generated centre by metabelian group with centre $\zeta(H)$ given by

$$\zeta(H) = \begin{pmatrix} 1 & 0 & R \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $\zeta(H)$ can be identified with R. We write $\bar{r}$ for the image in $\zeta(H)$ of an element $r \in R$ and similarly for subsets of R.

Lemma 3.4. (i) *If N is a normal subgroup of finite index in H then there is an ideal I of finite index in R such that $\bar{I} \leq N$.*

(ii) *If I is an ideal of R then $\bar{I}$ is closed in the profinite topology on H.*

(iii) *Let k,l be non-negative integers, and let I be the ideal $(x^{p^{k+1}} - 1)R + p^lR$ of R. Then $\bar{I}$ is closed in the pro-p topology on H.*

Proof. (i) We define

$$J_1 = \{r \in R \mid 1 + re_{12} \in N\zeta(H)\}, \qquad J_2 = \{r \in R \mid 1 + re_{23} \in N\zeta(H)\}$$

and

$$J = J_1 \cap J_2 ,$$

where as usual e_{ij} denotes the matrix with (i, j)-entry 1 and all other entries zero. Thus each of J_1, J_2 and J has finite index in R. Let V be the image of N under the obvious homomorphism from H to U, and let S be the subring of R generated by V. So V has finite index in U, the ring S is Noetherian, and R is finitely generated, by $h_1, \ldots, h_n$, say, as an S-module. Consideration of conjugation by elements of N shows that each of J_1, J_2 is an S-submodule, and hence so is J. Since

$$[1 = re_{12}, 1 + r'e_{23}] = 1 + rr'e_{13}$$

for $r, r' \in J$ we have $\overline{J^2} \leq N$. Now J/J^2 is a finitely generated module for the finite ring $S/(J \cap S)$, so that $J/J^2$is finite, and it follows that R/J^2 is finite. Define

$$I_k = \{r \in R \mid rh_k \in J^2\}$$

for $k = 1, \ldots, n$. Each I_k has finite index in R, being the kernel of the map $r \mapsto rh_k + J^2$ from R to R/J^2, and hence so has $I = \bigcap_{k=1}^{n} I_k$. Clearly I is an ideal of R with the property that $\bar{I} \leq N$.

For both parts (ii) and (iii) it is useful to note that $H/\zeta(H)$ is isomorphic to a restricted wreath product of a free abelian group of rank $2n$ by an infinite cyclic group, and is therefore residually a finite p-group. Thus $\zeta(H)$ is closed in the pro-p topology on H and *a fortiori* in the profinite topology.

(ii) Let K denote the kernel of the natural homomorphism $H \to \mathrm{GL}_3(R/I)$. Since R/I is a finite generated commutative ring, $\mathrm{GL}_3(R/I)$ is residually finite, and hence K is closed in the profinite topology on H. It follows that $\bar{I} = K \cap \zeta(H)$ is closed.

(iii) In this case, the image of H in $\mathrm{GL}_3(R/I)$ is a finite p-group and it follows by an argument similar to (ii) that $\bar{I}$ is closed in the pro-p topology on H.

This completes the proof of Lemma 3.4. □

Let B be a free abelian group of rank n. Our examples are obtained by making a careful choice of group homomorphism $\varphi : R \to B$ and setting

$$G = (H \times B)/\Delta , \quad \text{where } \Delta = \{(\bar{r}, \varphi(r)) \mid r \in R\} .$$

Let $B \geq B_0 \geq B_1 \geq B_2 \geq B_3 \geq \cdots$ be a descending chain of subgroups of finite index in B. We use additive notation for the group operation in B. Suppose that C is a further subgroup of finite index with $C \leq B_0$. We shall need to ensure that the chain of B_i satisfies various conditions, and we record the fact that an appropriate choice can be made next.

Lemma 3.5. *Let p and q be primes (which need not be distinct). Then for suitable* *choices of n, the subgroup C and the chain of B_i can be chosen so that the following hold*:

(i) B/B_0 *is an elementary abelian q-group of rank n*;

(ii) $B_k + C = B_0$, *for all* $k \geq 0$;

(iii) $B_k \not\leq C$, *for all* $k \geq 0$;

(iv) B_0/B_k *is a p-group, for all* $k \geq 0$;

(v) B_0/C *has order* q;

(vi) $\bigcap_{k \geq 0} B_k = 0$.

Proof. It suffices to choose a rank n free abelian group B_0 and subgroups B_i and C so that (ii)–(vi) are satisfied. Then B_0 can be embedded in B so that (i) holds.

If p and q are distinct then the situation is very simple: take $n = 1$ so that B_0 is infinite cyclic, let B_k be the subgroup of index p^k and let C be the subgroup of index q. We leave it to the reader to confirm that (ii)–(vi) are satisfied in this case.

Now suppose that $p = q$. Take $n = 2$ and regard B_0 as a rank-2 free abelian subgroup of the additive group $\mathbb{Z}_p$ of p-adic integers which contains 1. Let $B_k = p^k\mathbb{Z}_p \cap B_0$ for $k \geq 0$. Choose C to be a subgroup of index p in B_0 such that $B_1 + C = B_0$. Now (iv)–(vi) are clearly satisfied, and it is easy to check (ii). Finally, (iii) is implied by (ii) because C is a proper subgroup of B_0. □

In the following lemma we consider additive homomorphisms from the ring A to B.

Lemma 3.6. *There is a family of homomorphisms $(\varphi_i)_{i \in \mathbb{Z}}$ from A to B with the following properties*:

(i) φ_0 *is the trivial homomorphism*;

(ii) *for each* $k \geq 0$, φ_{p^k} *maps A isomorphically to* B_k;

(iii) *if i is not a power of p then the image of φ_i is contained in* C;

(iv) *if $k \geq 0$ and $i \equiv j \bmod p^{k+1}$ then $\varphi_i \equiv \varphi_j \bmod B_k$.*

Proof. We define φ_i by induction on $|i|$, beginning with $\varphi_0 = 0$. Suppose that h is a positive integer and that φ_i has been defined for all i with $|i| < h$ in such a way that (i)–(iv) hold. We now define φ_h and φ_{-h}.

Let m be the largest integer such that $p^m \mid (h - i)$ for some i with $|i| < h$, and for definiteness let l be the least i which witnesses this. If $h = p^k$ for some $k \geq 0$ then choose φ_h to be any homomorphism which satisfies (ii), and in all other cases set φ_h equal to φ_l. Since l is necessarily non-positive the image of φ_l is contained in C, and hence when h is not a power of p, the image of φ_h is contained in C.

Now we set φ_{-h} equal to $\varphi_{-l} + \theta$ where $\theta : A \to B$ is chosen as follows. We recall that the ring A is the direct sum of n copies of $\mathbb{Z}$; let e_r be the generator of the rth direct summand for $1 \leq r \leq n$. Since $B_m + C = B$, we can write $\varphi_{-l}(e_r) = b_r + c_r$ with $b_r \in B_m$ and $c_r \in C$, for each r with $1 \leq r \leq n$. Let θ be the map defined by $\theta(e_r) = -b_r$. Thus the image of θ is contained in B_m and the image of $\varphi_{-h} = \varphi_{-l} + \theta$ is contained in C.

It remains to check that (iv) holds for $i = \pm h$ and any integer j lying between $-h$ and h. We consider three cases:

Case 1. $i = h$ *and* $|j| < h$. Suppose that $h \equiv j \bmod p^{s+1}$. By choice of m and l, it follows that $s + 1 \leq m$ and hence $i \equiv l \bmod p^{s+1}$. By induction, $\varphi_i \equiv \varphi_l \bmod B_s$. If h is not a power of p then $\varphi_h = \varphi_l$ by definition, and hence (iv) holds. If h is a power of p then $l = 0$ and $h = p^m$. In this case $\varphi_h \equiv 0 \bmod B_s$ and again (iv) holds.

Case 2. $j = -h$ *and* $|i| < h$. Suppose that $i \equiv -h \bmod p^{s+1}$. By choice of m and l, it follows that $s + 1 \leq m$ and $i \equiv -l \bmod p^{s+1}$. By induction $\varphi_i \equiv \varphi_{-l} \bmod B_s$. Also, by definition, $\varphi_{-h} = \varphi_{-l} + \theta$ where θ has image in B_m. Since $B_m \leq B_s$ it follows that $\theta \equiv 0 \bmod B_s$, and now it follows that $\varphi_{-h} \equiv \varphi_{-l} \equiv \varphi_i \bmod B_s$, as required by (iv).

Case 3. $i = h$ *and* $j = -h$. If $h \equiv -h \bmod p^{s+1}$ then p^{s+1} divides $2h$, so if p is odd then p^{s+1} divides h. In any case, p^s divides h, whether or not p is odd. We can reduce to one of the cases already considered provided that there is an i with $|i| < h$ such that $i \equiv h \bmod p^{s+1}$. Thus the only case we need to consider is when $p = 2$ and $h = 2^s$. In this case $m = s$ and $l = 0$ and the result follows from the definitions of φ_h and φ_{-h}. □

Using the φ_i constructed above, we define an additive homomorphism φ from R to B by $\varphi(ax^i) = \varphi_i(a)$, for $a \in A$ and $i \in \mathbb{Z}$. Through the identification of R with $\zeta(H)$, there are two natural topologies on R which we need to consider: there is the profinite ideal topology on R, which by Lemma 3.4(i) and (ii) coincides with the subspace topology on $\zeta(H)$ induced by the profinite topology on H, and there is the topology on R which coincides with the subspace topology on $\zeta(H)$ induced by the pro-p topology on H. We shall denote these topologies by τ and τ_p respectively. The following general result can be applied in either case.

Lemma 3.7. *Let R be any commutative topological ring in which there is a base of open neighbourhoods of* 0 *consisting of ideals, and let F be a discrete abelian group. If φ is an additive homomorphism from R to F then the following are equivalent*:

(i) *φ is continuous*;

(ii) *the kernel of φ contains an open ideal.*

Proof. (i)$\Rightarrow$(ii) If φ is continuous then its kernel $\varphi^{-1}(0)$ is open and so it contains an open ideal.

(ii)$\Rightarrow$(i) If $\ker \varphi$ contains an open ideal I then for each $f \in F$, $\varphi^{-1}(f)$ is a union of cosets of I and so is open. Thus φ is continuous. □

Now we return to the specific ring $R = A[x, x^{-1}]$.

Lemma 3.8. *If R is viewed as a topological ring with either of the topologies τ or τ_p then the following hold:*

(i) *For each $k \geq 0$ the composite $R \to B \to B/B_k$ of φ with the natural projection of B onto B/B_k is continuous.*

(ii) *The composite $R \to B \to B/C$ is discontinuous.*

Proof. (i) Since every τ_p-open set is τ-open, it suffices to check continuity with respect to τ_p. Fix any $k \geq 0$, and let p^l be the exponent of B/B_k. Let I be the ideal of R generated by $x^{p^{k+1}} - 1$ and p^l. Lemma 3.4(iii) shows that $\bar{I}$ is closed in the pro-p topology on H, and hence I is τ_p-closed. Since I has finite index in R it is also τ_p-open. Lemma 3.6 guarantees that I lies in the kernel of the map $R \to B/B_k$ and hence this map is continuous by Lemma 3.7.

(ii) Here, it suffices to consider the topology τ. Let ψ denote the composite $R \to B \to B/C$, and suppose that ψ is continuous. By Lemma 3.7, ker ψ contains an ideal I of finite index in R. Let l be the order of the finite ring R/I. Choose any natural number k such that $l < p^k$. Since φ_{p^k} maps A isomorphically onto B_k and since B_k is not contained in C there exists an element a in A such that $\varphi_{p^k}(a) \in B_k \backslash C$, and hence

$$(*) \qquad \varphi(ax^{p^k}) \in B_k \backslash C\,.$$

Now, with a and k fixed, consider the sequence ax^j of elements of R where j varies between 0 and l. There are $l+1$ of these elements and since R/I has only l elements, there exist $j_1 < j_2$ with

$$ax^{j_1} \equiv ax^{j_2} \mod I\,.$$

Since I is an ideal, we deduce that

$$ax^{p^k} \equiv ax^{p^k + j_2 - j_1} \mod I\,,$$

and since $I \leq \ker \psi$ it follows that

$$(**) \qquad \varphi(ax^{p^k}) \equiv \varphi(ax^{p^k + j_2 - j_1}) \mod C\,.$$

However, $p^k < p^k + j_2 - j_1 \leq p^k + l < p^{k+1}$, and this implies that $\varphi(ax^{p^k + j_2 - j_1}) = \varphi_{p^k + j_2 - j_1}(a) \in C$, so by virtue of $(**)$ we must also have $\varphi_{p^k}(a) = \varphi(ax^{p^k}) \in C$, in contradiction to $(*)$. This completes the proof. $\square$

Theorem 3.9. *Let G be the quotient group $(H \times B)/\Delta$ where*

$$\Delta = \{(\bar{r}, \varphi(r)) \mid h \in H\}\,.$$

Then the following hold:

(i) *G is torsion-free, finitely generated, and centre by metabelian.*

(ii) *G is residually finite, and if $p = q$ then G is residually a finite p-group.*

(iii) *The profinite completion of G contains an element of order q, and if $p = q$ then the pro-p completion of G contains an element of order p.*

Proof. (i) First note that $G/\zeta(G)$ is isomorphic to $H/\zeta(H)$ and so is torsion-free and metabelian. Since $\zeta(G)$ is naturally isomorphic to B it follows that G is torsion-free and centre by metabelian. Finally G is finitely generated since both H and B are.

(ii) Note that $H \times B$ is residually a finite p-group. We need to check that Δ is closed in the profinite topology, and that if $p = q$ then it is closed in the pro-p topology.

Clearly $\zeta(H) \times B$ is closed in the profinite topology on H, and moreover the subgroup $\overline{\varphi^{-1}(B_k)} \times B_k$ is closed for each k by Lemma 3.8(i). Let $(\bar{r}, b)$ denote a typical element of the closure of Δ. If this does not belong to D then $b - \varphi r$ is a non-zero element of B. Hence by Lemma 3.5(vi) there is a k such that $b - \varphi r \not\in B_k$. It is now easy to see that $(\bar{r}, b)$ is not in $(\overline{\varphi^{-1}(B_k)} \times B_k)\Delta$, a contradiction.

(iii) We can regard the natural map $B \to G$ as identifying B isomorphically with a central subgroup of G. Lemma 3.8(ii) shows that C is not closed. Since B_0^q is contained in and has finite index in C, it follows that this subgroup is again not closed. On the other hand, B_0 is closed. Therefore there exists $b \in B$ such that b^q belongs to the closure of B_0^q but not to B_0^q itself. Let b_j be a sequence of elements of B_0 such that b_j^q converges to b^q. Passing to a subsequence of the b_j we may assume that the sequence converges in the profinite completion of G to an element b_∞. Evidently $b^{-1}b_\infty$ has order q, and the proof of Theorem 3.9 is complete. □

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